

Emergence of ripples on the surface of icicles

Antony Szu-Han Chen and Stephen W. Morris

Department of Physics, University of Toronto, 60 St. George St., Toronto, Ontario, Canada M5S 1A7

(Dated: January 22, 2013)

Natural icicles often exhibit ripples about their circumference which are due to a morphological instability. We present an experimental study that exposes the mechanism of the instability, using laboratory-grown icicles. We find that, contrary to theoretical expectations, icicles grown from pure water do not exhibit ripples. The addition of non-ionic surfactants that reduce the surface tension does not produce ripples. Instead, ripples emerge on icicles grown from water with dissolved ionic impurities. We find that even minute levels of impurity are sufficient to trigger ripples and that the growth rate of the ripples increases very weakly with ionic concentration. We outline a proposed theory of the instability based on constitutional undercooling.

Consider the icicle. Its elegant and familiar form is the result of a subtle interplay between the solidification dynamics of ice [1, 2] and the gravity-driven flow of the liquid water film flowing over its evolving surface [3–10]. The latent heat released by freezing, which controls ice growth, is advected by the water film and ultimately carried away by the surrounding sub-zero air, which is also flowing [4, 11]. Ideal, *platonic* icicles have been predicted to adopt a universal self-similar shape, independent of growing conditions [4]. In reality, natural [12, 13] and even laboratory-grown [14–16] icicles often deviate in interesting ways from the platonic ideal. The most striking of these deviations is the formation of trains of uniform ripples around the circumference of the icicle [5–10, 12–16]. Like the overall icicle shape, the wavelength of the ripples is remarkably independent of the growing conditions. Ripples have been observed to move slowly in the direction opposite to the water flow during growth [10, 14, 16].

Here, we present an experimental study of the physical parameters that influence the formation of icicle ripples that reveals the essential but hitherto unsuspected role of dissolved impurities in their formation. We find that ripple growth rates increase roughly logarithmically with the concentration of ionic impurities in the feed water. Pure water icicles grow no ripples. Dissolved air does not, by itself, cause ripples to grow. Reducing the air-water surface tension with a strong non-ionic surfactant does not produce ripples unless a dissolved ionic impurity is also present. These results are in direct contradiction to existing theories [5–10] and may find their explanation in a theory which incorporates the advection-diffusion of both heat and salt, with constitutional undercooling. We outline this proposed theory here, which has some similarity to a recent theory of crenulations on stalactites [17].

In addition to ambient temperature and feed water flow rate [14], it has previously been shown that water purity and air motion are important to the overall icicle morphology [16]. Icicles grown in still air tend to develop multiple tips, while those grown from distilled water are significantly smoother and closer to the self-similar shape predicted by theory [4] than those grown from untreated

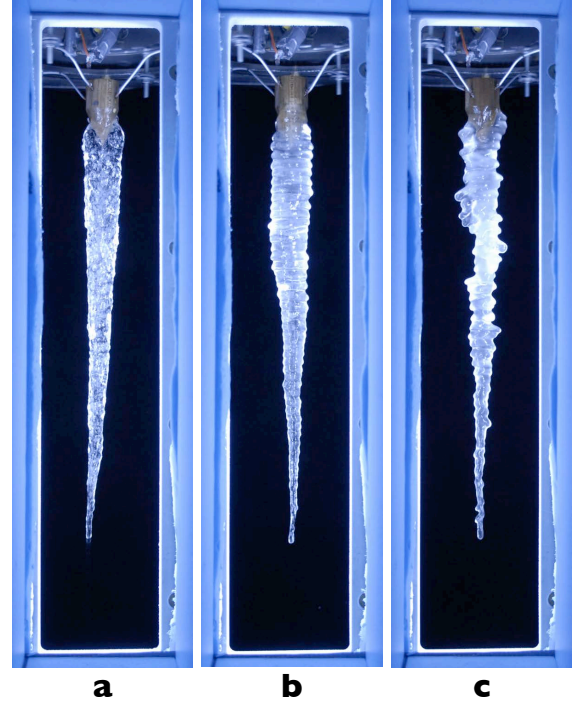


FIG. 1: (Color) Images of three icicles grown under identical conditions of ambient temperature, water supply rate, and nozzle temperature. (a) was made with distilled water only; (b) was made with distilled water plus NaCl with a mass fraction of $(8.0 \pm 0.2) \times 10^{-5}$; (c) was made with distilled water plus NaCl with a mass fraction of $(1.278 \pm 0.002) \times 10^{-3}$. See Supplemental Material at [URL will be inserted by publisher] for time lapse videos of the growth process.

tap water. Tap water icicles show prominent ripples and other non-ideal morphologies [16]. Here, we focus on comparing icicles grown under the same conditions of temperature, water supply rate, and air motion, varying only the composition of the feed water. We add small quantities of salt to distilled water to achieve levels of impurities similar to tap water; we also investigate both the role of dissolved gas and the effect of surfactants which reduce the air-water surface tension. We find that the presence of small amounts of ionic impurities is crucial

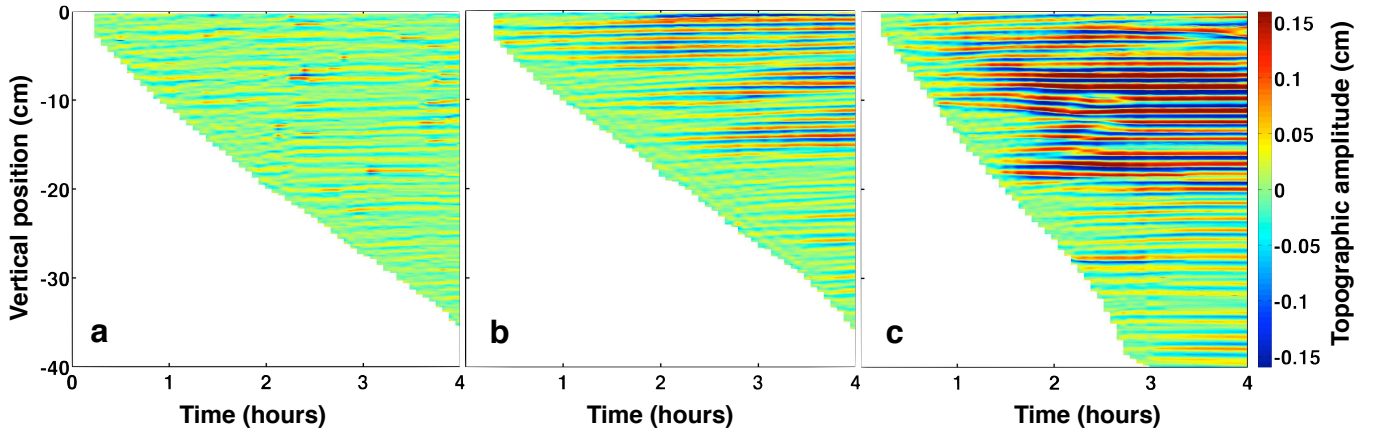


FIG. 2: (Color) Spacetime plots showing the time evolution of the ripple topography, viewed from one fixed rotational position, for the three icicles shown in Fig. 1. The vertical coordinate is the position on the icicle measured from the root down; the horizontal coordinate is time. The edge of the white region shows the downward progress of the tip of the growing icicle.

to the formation of ripples.

The apparatus was previously described in Ref. [16]. Icicles were grown below a sharpened wooden support suspended inside an insulated, refrigerated box. The support was rotated, at a speed of 4 minutes per revolution, to encourage axisymmetry and to allow all sides of the icicle to be imaged. The air was gently stirred by internal fans in the corners of the box. The wall temperature was controlled by a commercial bath. The feed water supply was delivered by a peristaltic pump to a temperature-controlled nozzle that was slightly off-axis from the rotating support, in order to distribute the water evenly. The rotational position of the support was indexed so that eight reproducible views of the icicle could be imaged on each rotation.

We report data from 16 icicles using feed water with varying amounts of sodium chloride and surfactant, grown under otherwise identical conditions: an ambient wall temperature of -12.2 ± 0.3 °C, a feed water temperature of 3.0 ± 0.3 °C, and a water supply rate of 2.0 ± 0.1 g/min. Even from direct inspection, there are obvious qualitative differences in the form of the resulting icicles; three examples are shown in Fig. 1. Icicles made with distilled water have no significant ripples and a smooth, carrot-like form that is close to the predicted self-similar shape [4, 16]. As the salt concentration is increased, ripples appear at concentrations similar to, or even below, the ionic impurity levels of Toronto tap water. In addition to the appearance of the ripples, the overall form of the icicle also deviates further from self-similarity and axisymmetry with increasing salt concentration [16].

We used edge detection to map the topography of the icicle in high-resolution digital images, taken at a rate of eight images per four-minute icicle rotation. To extract the amplitude and wavelength of the ripples as a function of time, this data was de-trended using Fourier techniques to remove the overall background shape. The

data acquisition and analysis are discussed further in the Supplemental Material at [URL will be inserted by publisher].

The resulting de-trended profiles of the three example icicles from Fig. 1 are shown in the spacetime plots in Fig. 2. Here, the right hand edge of each icicle is shown at a particular rotational position. The ripples appear in a patchy way and generally grow to a maximum amplitude early in the lifetime of the icicle. The small residual topographic features on the distilled water icicle that survive the de-trending filter are consistent with the noise floor of the amplitude measurement, while the largest features on the saltiest icicle grow to a few mm in amplitude. A trend toward very slow upward motion of the ripples is just barely visible in Fig. 2. The overall velocity of this motion is consistent with previous observations of ripples growing on vertical cylinders [16] and inclined troughs [10]. The generally upward crest motion is sometimes locally interrupted by the appearance of new ripples, ripple mergers, and other dynamics.

Fig. 3 shows ripple amplitudes *vs.* time for icicles made from various solutions. These were obtained by fitting the peak in the spatial Fourier spectrum of the de-trended icicle topography, then averaging over one rotation and over repeated runs under the same conditions. We considered ripples only in the uppermost 10 cm of the icicle. Ripples farther down the icicle have had less time to grow and may grow under saltier conditions due to the exclusion of salt by ice formation higher up. In Fig. 3, time zero corresponds to the time when the icicle reaches 10 cm in length. The statistical error bars, which become larger for saltier icicles, mainly reflect the progressive loss of axisymmetry as the salt concentration increases. Distilled water, aerated distilled water, and distilled water with Triton X-100 surfactant all show no measurable amplitude increase. The ripple amplitudes for salty icicles increase approximately linearly for the first 1.5 hours of

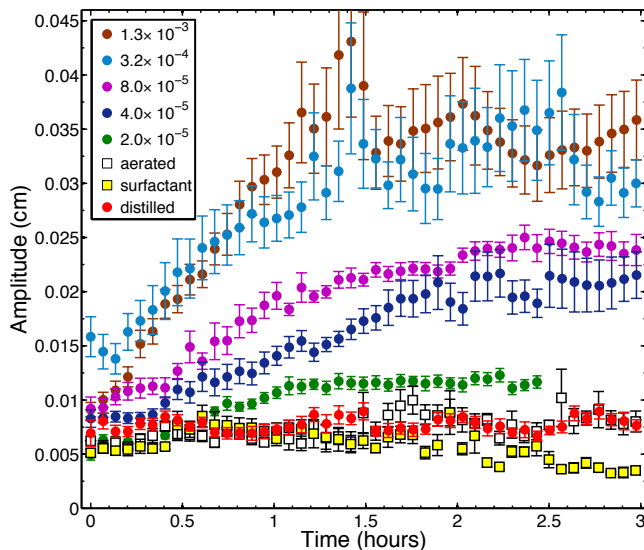


FIG. 3: (Color) Time-series of the amplitude of ripples on icicles made with a variety of salty feed waters. The mass fraction of NaCl is shown in the legend. Also shown are pure distilled water, aerated distilled water, and distilled water with surfactant. For icicles grown from salt water, both the ripple growth rate and the maximum amplitude reached increase with salt concentration. Distilled water without added salt shows no significant ripple amplitude growth. The error bars indicate the standard error of the mean of rotationally averaged data.

icicle growth. The long-term nonlinear behavior of the ripple amplitude is more complex. In many cases, the amplitude simply saturates at its maximum value, while in others it decays with time. In some cases, ripples continue growing slowly until the icicle reaches the maximum length allowed by the apparatus.

Fig. 4 shows some characteristics of the icicle ripples *vs.* the salt concentration of the feed water. In Fig. 4(a), the growth rate of the ripple amplitude is given by the slope of Fig. 3(a) for the first 1.5 hours. In Fig. 4(b), we plot the error-weighted mean of the ripple wavelength time-series after 1.5 hours of growth. As the salt concentration is increased, the ripples grow faster. The growth rate increases only approximately logarithmically with the salt concentration. Measurable ripples appear at the remarkably low concentration of 10^{-5} weight % salt, *i.e.* only 10 mg of salt per liter of water. The ripple wavelength was 1.04 ± 0.04 cm, independent of the salt concentration.

We also grew icicles from distilled water that had had air bubbled through it for 12 hours, which may therefore be assumed to be saturated, or possibly supersaturated, with dissolved gases. In fact, all of the solutions used in this experiment had long been exposed to air and presumably contained some concentration of dissolved gas. As shown in Fig. 3(a), ripple formation from deliberately aerated distilled water was not significantly different from that of untreated distilled water — no significant ripples

formed. We conclude that dissolved gases alone are insufficient to trigger the rippling instability. Moreover, since the air-water surface tension of the thin water film flowing over the icicle surface has featured prominently in theories of ripple formation [5–10, 17], we grew icicles from distilled water with an added non-ionic surfactant, Triton X-100. The results are shown in Fig. 3. Icicles grown from distilled water with sufficient surfactant to reduce the surface tension by 47% showed no ripples. Details on the composition and preparation of the water samples is given in the Supplemental Material at [URL will be inserted by publisher].

Models of icicles [3–10] assume that their surface is covered by a thin film of flowing water which has a parabolic shear flow profile with a surface speed U . Film thicknesses h are typically $100 \mu\text{m}$ or less, with a Reynolds number $\text{Re} = Uh/\nu \sim 1$, where ν is the kinematic viscosity. Any topography present on the ice surface is reflected in the shape of the water-air surface in a manner controlled by the surface tension γ , which appears in the free surface boundary conditions in the form of the Weber number $\text{We} = \rho U^2/\gamma \sim 10^{-4}$, where ρ is the density. Because the thermal conductivity of air is much less than that of water, the temperature difference across the thin film ΔT is much smaller than the total temperature difference driving the growth [8, 11]. In our experiments, the latter was 12.2°C , while $\Delta T < 0.1^\circ \text{C}$.

The latent heat released at the ice-water interface is diffused and advected through the flowing water film and carried away by the surrounding stirred air. The relevant dimensionless parameter in the thermal advection-diffusion equation for the water film is the Péclet number $\text{Pe}_h = Uh/\kappa \sim 5$, where κ is the thermal diffusivity. It has been shown that, in addition to diffusion and advection, both thermal radiation and water evaporation at the water-air interface make significant contributions to the heat transport through the surrounding air [11].

The rate of ice growth V is contained in the Stéfán condition [1, 2], which is a boundary condition on the temperature gradient ∇T at the ice-water interface. The Stéfán boundary condition depends on the Stéfán number $\text{St} = L/C_P \Delta T \sim 1300$, where L is the latent heat of fusion and C_P is the heat capacity. $\text{St} \gg 1$, indicating that ice growth is in the quasi-steady limit [1] where kinetic undercooling effects should be very small.

Existing theories of icicle ripples [5–10] have not considered the effect of impurities, and included only thermal and surface tension effects. They predict ripples on pure water icicles, which we do not observe. The linear stability analysis by Ueno [8] found an instability to growing ripples that travel against the direction of flow. As is usual for such problems, the devil is in the boundary conditions. Ueno assumed [10] that the ice-water interfacial temperature obeyed the Stéfán condition on ∇T , but that T itself could deviate by δT away from the equilibrium melting temperature T_m , which he attributed to

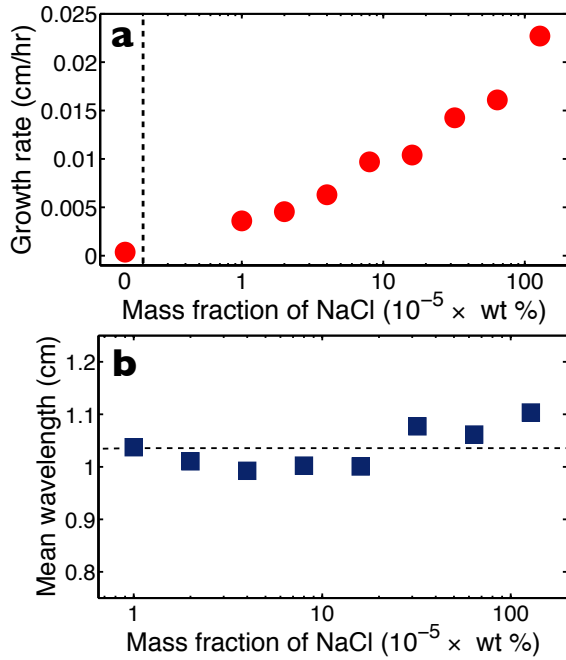


FIG. 4: Characteristics of icicle ripples: (a) the amplitude growth rate during the first hour; (b) the mean wavelength after one hour. Each is plotted against the mass fraction of salt in the feed water on a logarithmic scale. Data to the left of the break in (a) is for distilled water with no added salt.

kinetic undercooling [1, 2, 10]. In this theory, the predicted wavenumber k_c and the amplitude growth rate σ_r of the most unstable mode are given by [8]

$$k_c = \left[\frac{3 \rho g}{\text{Pe}_h \gamma h} \right]^{1/3}, \quad \sigma_r(k_c) = \frac{3}{4} V k_c \sim \gamma^{-1/3}, \quad (1)$$

where g is the acceleration due to gravity. Thus, an increase in surface tension is predicted to result in a weak decrease in the ripple growth rate. Over the range of salt concentrations in Fig. 4(a), however, the air-water surface tension increases by 0.04% [18], while the observed growth rate increases by a factor of 6. Our results contradict the $\gamma^{-1/3}$ dependence of Eqn. 1. Furthermore, the addition of surfactant, which lowers the air-water surface tension, produced no measurable ripples and hence no effect on their growth rate. Thus, our results are broadly inconsistent with existing theories [5–10].

We now present some general ideas on how to account for the ripple growth on salty icicles [19]. In addition to the temperature field, the presence of salt in the water film adds another advection-diffusion equation for the salt concentration C and a second, much larger Péclet number $\text{Pe}_s = Uh/D \sim 400$, where D is the molecular diffusivity of salt. Then, a boundary condition similar to the Stéfán condition links the production of salt excluded by the advancing ice to ∇C at the ice-water interface. This causes a build-up of salt near the interface. This, in turn, produces constitutional undercooling [2], so that

the temperature at the ice-water interface is given by $T = T_m - mC$, where m is the slope of the liquidus line, which accounts for equilibrium freezing point depression. This theory ignores kinetic undercooling effects, which might still be present for nonzero V . Thus, mC plays the role of δT in Ueno’s theory [10], but the physical mechanism of the temperature deviation is different, and the salt concentration must be non-zero for ripple formation. In addition, the detailed effect of the salt depends on Pe_s .

A natural scaling of the fully double-diffusive problem with constitutional undercooling introduces another dimensionless number, $M = mC_0/\Delta T$, where C_0 is the concentration of salt in the feed water. M links the boundary conditions on T and C at the ice-water interface. The surprising sensitivity of the ripple instability to very small concentrations of salt may be because ΔT is small, only a tiny C_0 is enough to make $M \sim 1$. The absence of ripples on pure water icicles presumably reflects the fact that their growth rate is small, or possibly even negative, for $M = 0$. It remains to make this idea quantitative.

There is an imperfect analogy between icicles and stalactites [17, 20]. Stalactites are formed by the deposition of CaCO_3 under similar thin film flow conditions. A theory of ripples, known as *crenulations*, on stalactites has recently been proposed [17]. Crenulations also involve several advecting and diffusing species, with $[\text{CO}_2]$ playing a similar role to the temperature in icicles. The dynamics, however, is made more complicated by bulk chemical reactions that have no analog in salty icicles.

In conclusion, we have shown that small amounts of ionic impurities are required for the formation of ripples on icicles and that ripples are not present on icicles grown from pure water, even with dissolved gases. Ripples are not produced by simply reducing the surface tension of pure water with surfactants. Ripple growth rates show a very weak, roughly logarithmic dependence on salt concentration. We have outlined a proposed new theory of the rippling instability based on constitutional undercooling.

We thank J. Wettlaufer, J. Neufeld, M. G. Worster and R. E. Goldstein for insightful discussions, and C. Ward for his assistance with the surface tension measurement. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES FOLLOW AFTER SUPPLEMENTARY MATERIAL.

Supplementary Information for *Physical Review Letters* Emergence of ripples on the surface of icicles

Time lapse video of growing rotating icicles

A supplementary video available on [YouTube] or [Flickr]. This supplementary video shows the evolution of the three icicles shown in Fig. 1 in time lapse. Refer to the caption of Fig. 1 for the detailed growing conditions.

Water Samples

Table I shows the measured physical parameters of the various water samples used. Toronto tap water is shown for comparison. The distilled water was supplied in bulk by Canadian Springs TM. Note that, while it is much purer than tap water, it is not near the limits of what purity can be achieved. Nevertheless, this level of purity was sufficient to suppress the rippling instability. The added NaCl was ACS reagent grade, supplied by Sigma-Aldrich. The added non-ionic surfactant was Triton X-100, which is t-Oct-C₆H₄-(OCH₂CH₂)_xOH, $x = 9-10$. The Triton X-100 was also supplied by Sigma-Aldrich.

The conductivity and composition of the samples were analyzed by the ANALEST lab, Department of Chemistry, University of Toronto. Metal ion concentrations were determined by inductively coupled plasma atomic emission spectrometry, using Perkin Elmer Optima 7300DV. Anions were identified by ion chromatography, using Perkin Elmer Series 200, Alltech Eris 1000HP, Alltech 550 Conductivity Detector, and Phenomenex Star Ion 300 Column. Conductivities were measured using an Alltech 550 Conductivity Detector. The surface tensions of distilled water, tap water, and surfactant solutions were measured using the capillary tube method at room temperature. The surface tensions of salt solutions were calculated using Ref. [18]. The results are shown in Table I.

Temperature, humidity, and air flow measurements

The apparatus is a slightly improved version of the one discussed in Ref. [16]. The temperature of the inlet nozzle was measured with a thermocouple and feedback controlled by means of a computer-controlled heater. It was maintained at 3.0 ± 0.3 °C to prevent freezing of the inlet pipe. The temperature of the feed water had no important effect on the icicle growth. The rotating support upon which the icicle grew was suspended by stiff wires and not actively cooled. All of the heat from the growing icicle was transported by advection and diffusion through the surrounding air, by radiation, or by evaporation /

condensation of water [11]. The latter mechanism depends on the relative humidity, which was continuously monitored during the experiment. For all the data reported here, the relative humidity was 87 ± 1 %.

The air inside the icicle-growing box was stirred by eight small computer fans: four at the top corners pointed straight down, and four at the bottom corners pointed straight up. Air motion affects the overall growth rates of icicles [14] and is also necessary to suppress the tendency of icicles to grow multiple tips [16]. Using a digital anemometer (Omega HHF92A), we determined the total air flux from each fan to be $(1.59 \pm 0.05) \times 10^{-2}$ m³/s. Dividing the volume of the box by eight times this flux gives a characteristic time of 0.75 ± 0.05 s — this is an estimate of the time it takes for the fans to circulate all of the air enclosed in the box once. Thus, the air in the box may be regarded as well-stirred, save for a thin viscous and thermal boundary layer near the icicle and near the walls of the box. Thermocouple measurements of the air in the well-stirred region show a mean temperature of -9.7 ± 0.2 °C, a few degrees warmer than the walls of the box, which are controlled at -12.2 ± 0.3 °C by a circulating bath.

Image acquisition and analysis

To investigate the evolution of icicle ripples, we analyzed time series of high-resolution digital images taken during the experiment. MATLAB's Sobel algorithm was used to detect the left- and righthand edges of the icicle in each image. To extract the ripple wavelength and amplitude, the following algorithm was used to de-trend this topographic data: the data was first Fourier transformed, and the region in the power spectrum corresponding to a wavelength between 0.5 - 1.5 cm was set to zero; upon back transforming, the resulting ripple-free filtered topography was smoothed with a moving boxcar average with a width of 0.5 cm, giving the overall background shape; finally, the background shape was subtracted from the original raw edge to obtain the de-trended ripples. While the entire de-trended icicle profile is shown in Fig. 2, only the topmost 10 cm of the icicle was included in the ripple amplitude and wavelength measurements.

The resolution of the digital images was approximately 0.018 cm per pixel, so each of the two edges of a 10 cm segment of the icicle contributed about 550 edge detection measurements. The Fourier power spectrum of the de-trended data was fit to a Gaussian peak to extract the amplitude and wavelength of the ripples. The data was averaged over both edges and one rotation consisting of 8 unique views, so each data point in Fig. 3 of the paper was based on averaging over 8800 edge detection measurements. In some cases, the data was further averaged over multiple runs under the same conditions. The noise floor, which is reached by the residual topog-

Solution	Main impurities	Concentration	Conductivity	Surface tension
Distilled water (DW)	Ca ⁺²	0.037 mg/L	2 μS/cm	0.072 N/m
	K ⁺	0.026 mg/L		
	Na ⁺	0.014 mg/L		
	Ba ⁺²	0.003 mg/L		
DW + Triton X-100	Triton X-100	200 mg/L	1.7 μS/cm	0.039 N/m
DW + NaCl	NaCl	80.0 mg/L	211 μS/cm	0.072 N/m
Toronto tap water	Ca ⁺²	37.5 mg/L	419 μS/cm	0.071 N/m
	Na ⁺	10.7 mg/L		
	Mg ⁺²	8.82 mg/L		
	K ⁺	1.66 mg/L		
	Si ⁺²	1.22 mg/L		
	Cu ⁺²	0.46 mg/L		
	SO ₄ ⁻²	31.60 mg/L		
	Cl ⁻	25.11 mg/L		
	NO ₃ ⁻	0.96 mg/L		
	F ⁻	0.41 mg/L		
	Zn ⁺² , V ⁺² , Ni ⁺² , Ba ⁺² , Fe ⁺² , Mn ⁺²	< 0.2 mg/L		

TABLE I: Measured compositions, conductivities, and surface tensions of the water samples. Toronto tap water, which produces icicles exhibiting prominent ripples, is shown for comparison.

Physical parameters		
Typical icicle radius	R	0.012 m
Volumetric flow rate	Q	$3.3 \times 10^{-8} \text{ m}^3/\text{s}$
Typical radial growth speed	$V = dR/dt$	$1.7 \times 10^{-6} \text{ m/s}$
Water film thickness	$h = (3Q\nu/2\pi gR)^{1/3}$	$6.2 \times 10^{-5} \text{ m}$
Surface speed of water film	$U = gh^2/2\nu$	0.011 m/s
Driving temperature difference	$T_m - T_{\text{wall}}$	12.2 $^{\circ}\text{C}$
Water film temperature difference	$\Delta T = \rho_{\text{ice}}LVh/\Lambda$	0.06 $^{\circ}\text{C}$
Liquidus slope for salt water	m	3.706 $^{\circ}\text{C kg/mol}$
Molality of salt	C_0	0.0219 mol/kg
Constitutional undercooling	$-mC_0$	-0.08 $^{\circ}\text{C}$
Latent heat of fusion	L	$3.34 \times 10^5 \text{ J/kg}$
Kinematic viscosity	ν	$1.8 \times 10^{-6} \text{ m}^2/\text{s}$
Density of water	ρ	$1.00 \times 10^3 \text{ kg/m}^3$
Specific heat of water	C_P	$4.198 \times 10^3 \text{ J/kgK}$
Thermal diffusivity of water	κ	$1.36 \times 10^{-7} \text{ m}^2/\text{s}$
Molecular diffusivity of salt	D	$1.6 \times 10^{-9} \text{ m}^2/\text{s}$
Surface tension of pure water	γ	0.0757 J/m ²
Dimensionless groups		
Ratio of growth to flow speeds	$v = V/U$	1.6×10^{-4}
Reynolds number of water film	$\text{Re} = Uh/\nu$	0.36
Weber number	$\text{We} = \rho U^2 h/\gamma$	9.1×10^{-5}
Péclet number for heat	$\text{Pe}_h = Uh/\kappa$	4.8
Péclet number for salt	$\text{Pe}_s = Uh/D$	410
Stéfan number	$\text{St} = L/C_P \Delta T$	1300
Constitutional undercooling number	$M = mc_0/\Delta T$	1.3

TABLE II: Typical quantities and some dimensionless groups for the case shown in Fig. 1(c). Here, $\Lambda = \kappa\rho C_P$ is the thermal conductivity of water, the values of ν , ρ , γ , κ , Λ , and C_P are from Ref. [18], the value of D is from Ref. [21], and R is taken after 2 hours of growth.

raphy of a distilled water icicle, was approximately 0.005 cm, or about 1/3 of a pixel. The averaged, best-fit ripple amplitudes, shown in Fig. 3, are smaller than the typical amplitudes within the patches of de-trended ripples shown in Fig. 2, because the fit included Fourier signals from regions with no ripples, which reduced the average. The errors, shown in Fig. 3, were estimated from the standard error of the averaging sample. The whole algorithm was tested on simulated images which had known ripple amplitudes and wavelengths.

Physical parameters and dimensionless numbers

In Table II, we list the important physical quantities likely to be involved in any theory of saline icicle ripples. We also estimate the relevant dimensionless groups discussed in the paper for a reference icicle – the case shown in Fig. 1(c) of the paper. The growth of this icicle is also shown in time lapse in the supplementary movie.

The water film thickness h can be calculated from the measured volumetric flow rate of the feed water Q , which is held constant. The film thickness can then be used to estimate the flow speed U at the surface of the water film, treated as having a parabolic flow profile. Most of the dimensionless quantities depend on combinations of h and U . It is important to note that, since the radius R of the icicle grows with time, h and U , as well as all the dimensionless quantities that depend on them, evolve significantly over the lifetime of the experiment.

To estimate ΔT , the temperature difference across the water film, we use the Stéfán condition and treat the film as having a perfect parabolic profile so that advection can be neglected. Then the heat flux is purely conductive and equal to $\Lambda \Delta T / h$. This must match the rate of heat production by ice growth $\rho_{\text{ice}} L V$. We use the measured ice growth speed V to estimate ΔT .

Since the free surface flow speed U of the parabolic profile is much greater than the typical radial ice growth speed V , a quasi-static approximation is justified, i.e.

the steady-state equations for the fluid flow and the advection-diffusion of heat and salt are solved over a slowly evolving ice topography, approximated as static.

-
- [1] G. K. Batchelor, H. K. Moffat, and M. G. Wortser, *Perspectives in Fluid Mechanics* (Cambridge, 2000).
 - [2] S. H. Davis, *Theory of Solidification* (Cambridge, 2001).
 - [3] L. Makkonen, *J. Glaciol.* **34**, 64 (1988).
 - [4] M. B. Short, J. C. Baygents, and R. E. Goldstein, *Phys. Fluids* **18**, 083101 (2006).
 - [5] N. Ogawa and Y. Furukawa, *Phys. Rev. E* **66**, 041202 (2002).
 - [6] K. Ueno, *Phys. Rev. E* **68**, 021603 (2003).
 - [7] K. Ueno, *Phys. Rev. E* **69**, 051604 (2004).
 - [8] K. Ueno, *Phys. Fluids* **19**, 093602 (2007).
 - [9] K. Ueno and M. Farzaneh, *Phys. Fluids* **22**, 017102 (2010).
 - [10] K. Ueno, M. Farzaneh, S. Yamaguchi, and H. Tsuji, *Fluid Dyn. Res.* **42**, 025508 (2010).
 - [11] J. A. Neufeld, R. E. Goldstein, and M. G. Worster, *J. Fluid Mech.* **647**, 287 (2010).
 - [12] H. Hatakeyama and S. Nemoto, *Geophys. Mag.* **28**, 479 (1958).
 - [13] N. Maeno and T. Takahashi, *Low Tem. Sci. Ser. A*, **43**, 139 (1984).
 - [14] N. Maeno, L. Makkonen, K. Nishimura, K. Kosugi, and T. Takahashi, *J. Glaciol.* **40**, 319 (1994).
 - [15] S. Matsuda, M.Sc. thesis, Hokkaido University, 1997.
 - [16] A. S. Chen and S. W. Morris, *Phys. Rev. E* **83**, 026307 (2011).
 - [17] C. Camporeale and L. Ridolfi, *Phys. Rev. Lett.* **108**, 238501 (2012).
 - [18] M. H. Sharqawy, V. J. H. Lienhard, and S. M. Zubair, *Desalin. Water Treat.* **16**, 354-380 (2010).
 - [19] S. W. Morris and A. S. Chen, in preparation.
 - [20] M. B. Short, J. C. Baygents, J. W. Beck, D. A. Stone, R. S. Toomey, and R. E. Goldstein, *Phys. Rev. Lett.* **94**, 018501 (2005).
 - [21] R. Riquelme, I. Lira, C. Pérez-López, R. A. Rayas, and R. Rodríguez-Vera, *J. Phys. D: Appl. Phys.* **40**, 2769 (2007).